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AN ELEMENTARY PROOF OF AN INFINITE DIMENSIONAL FATOU'S LEMMA WITH AN APPLICATION TO MARKET EQUILIBRIUM ANALYSIS

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Abstract

A Fatou's lemma for the Gel'fand integrable correspondences (multifunctions) will be proved. The result will be applied to analysis of a large exchange economy on the commodity space ℓ^{∞}. We define the economy as a measurable map from a measure space to the space of consumers' characteristics following Aumann [2, 4], and prove the existence of competitive equilibria.

1. Introduction

The classical Fatou's lemma for real valued functions have been extended to mappings with values of finite dimensional vectors by Aumann [3]; Hildenbrand [13]; and Schmeidler [25]. These studies were closely related with the research of a market model with a measure space of consumers.

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Aumann [2] defined an economy by a map $\succsim : A \to 2^{X \times X}$, which assigns each consumer $a \in A$ its preference and a map $\omega : A \to X$, which assigns a its initial endowment, where X is the non-negative orthant of the commodity space $\mathbb{R}^{\ell}$, which is identified as a consumption set and $(A, \mathcal{A}, \nu)$ is an atomless measure space. Every element $a \in A$ is interpreted as a "name" of a consumer, and each value of the map $(\succsim_a, \omega(a))$ is the characteristics of the consumer a (the individual form of the economy). The main result of the paper showed that the set of core allocations coincides with the set of allocations, which are supported as competitive equilibria (core equivalence theorem). Hildenbrand [13] established that the economy is defined by a measurable map $\mathcal{E} : A \to \mathcal{P} \times \Omega,\ a \mapsto \mathcal{E}(a) = (\succsim_a, \omega(a))$, where $\mathcal{P}$ is the set of preferences and Ω is the set of endowment vectors. Aumann [4] made the remarkable observation that for demonstrating the existence of the competitive equilibrium $(\boldsymbol{p}, \xi(a))$, where $\boldsymbol{p}$ is the equilibrium price vector and $\xi(a)$ is the consumption vector of the consumer $a \in A$, one does not have to assume the convexity on the preferences. This is a mathematical consequence of the Liapounoff theorem, which asserts that the range of a *finite dimensional* vector measure is convex. (See Diestel and Uhl [11], Chapter IX for details.)

A standard strategy for the proof of the existence of equilibria is that one truncates the consumption set by the cubes of the length $k = 1, 2, \ldots,$ defining $X_k = \{\xi \in X | \, \|\xi\| \leq k\}$. Then it is easy to show that the truncated sub-economy $\mathcal{E}_k$ obtained by replacing X with X_k has the equilibrium $(\boldsymbol{p}_k, \xi_k(a)),\ k = 1, 2, \ldots.$ Since we can normalize the price vectors $\|\boldsymbol{p}_k\| = 1$ for all k, we have $\boldsymbol{p}_k \to \boldsymbol{p}$ for some $p \in \mathbb{R}^{\ell}$. The Fatou's lemma in ℓ-dimension, $Ls(\int_A \xi_k(a) d\nu) \subset \int_A Ls(\xi_k(a)) d\nu$ (Fact 12 in Section 2), works here, and one obtains an integrable map $\xi(a)$ with $\xi(a) \in Ls(\xi_k(a))$ a.e. in A, where Ls means the limit set (see the next section). We can show that $(\boldsymbol{p}, \xi(a))$ is a competitive equilibrium for $\mathcal{E}$.

The economy with infinite time horizon, or more generally speaking, with infinitely many commodities was formulated by Bewley [6] in which the commodity space was defined by the space $\ell^{\infty} = \{\xi = (\xi^t) | \sup_{t \geq 1} |\xi^t| < +\infty\}$, the space of the bounded sequences with the supremum norm. We will see in the next section that on this commodity space, the space of all summable sequences, $\ell^1 = \{\boldsymbol{p} = (p^t) | \sum_{t=1}^{\infty} |p^t| < +\infty\}$, is a natural candidate of the price space. The value of a commodity $\xi = (\xi^t) \in \ell^{\infty}$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is then given by the natural "inner product" $\boldsymbol{p}\xi = \sum_{t=1}^{\infty} p^t \xi^t$. Bewley [6] established the existence of competitive equilibria for economies with finite number of consumers on this commodity space. Thereafter, this commodity space has been applied to theories of intertemporal resource allocations and capital accumulation by Bewley [7]; Yano [32]; Suzuki [26] among others; see also Suzuki [27].

The above results of Aumann and Bewley have been tried to be unified by several authors. For example, Bewley [8] and Noguchi [20] proved the equilibrium existence theorems for the economies with the measure space of consumers on the commodity space ℓ^{∞}. Khan and Yannelis [17] and Noguchi [19] proved the existence of a competitive equilibrium for the economies with the measure space of agents in which the commodity space is a separable Banach space, whose positive orthant has a norm interior point. Bewley worked with an exchange economy, and Noguchi [19, 20] proved his theorems for an economy with continua of consumers and producers[1].

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[1] On the other hand, Suzuki [28, 29] proved the existence and the core equivalence of competitive equilibria for an economy with a continuum of consumers, which is slightly different from those of the above literatures. He defined the economy as a probability measure μ on the set of agents' characteristics $\mathcal{P} \times \Omega$ (the coalitional form or the distribution form). Then the competitive equilibrium of this economy is also defined as a price vector $\boldsymbol{p} \in \ell^1$ and a probability measure ν on $X \times \mathcal{P} \times \Omega$. These definitions of the economy and the competitive equilibrium on it were first proposed by Hart and Kohlberg [12], and applied to the model on the space $ca(K)$ by Mas-Colell [18] (the model of the commodity differentiation. Bewley [8] also used this approach).

This time, one needs the Fatou's lemma on the infinite dimensional spaces. We start with an economy with a bounded consumption set. This is a standard assumption. See Bewley [7]; Khan and Yannelis [17]; Noguchi [19, 20]; and Suzuki [28, 29][2]. Usually the proof is carried out by approximating the large-infinite dimensional economy by large-finite dimensional sub-economies, a technique, which we will also utilize in this paper. In the course of the approximation, one is expected to apply the Fatou's lemma. Balder and Sambucini [5]; Cornet and Mèdècin [9]; and Yannelis [30] have proved the infinite dimensional versions of the Fatou's lemma. All of them, however, only ensure that the limit set of the integrals of finite dimensional allocations is contained in the integral of the closed convex hull of the limit set of the sequences. That is, $Ls(\int_A \xi_k(a)d\nu) \subset \int_A \overline{co}Ls(\xi_k(a))d\nu$, and one obtains the limit allocation with $\xi(a) \in \overline{co}Ls(\xi_k(a))$, where $\overline{co}$ means the closed convex hull. As a consequence, one has to assume the convexity of the preferences. Indeed, Khan and Yannelis, and Bewley assumed that the preferences are convex. Noguchi assumed that a commodity vector does not belong to the convex hull of its preferred set[3]. These assumptions obviously weaken the impact of the Aumann's classical result, which revealed the "convexfying effect" of large numbers of the economic agents. However, the convexity of preferences seems to be indispensable for proving the existence of equilibria for the "individual form" of the economy at the present stage of our knowledge. Indeed, we will apply a version of the Fatou's lemma and impose the convexity of preferences also in the present paper.

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[2] It seems too demanding to allow the possibility of unbounded consumptions for each individual in such a "huge" market with infinitely many consumers and commodities.

[3] An advantage of the "coalitional form" of Hart and Kohlberg is that it can dispense with the Fatou's lemma in the proof, and consequently, the convexity of preferences. Indeed, Mas-Colell [18] and Suzuki [28, 29] proved equilibrium existence theorems without the convexity of preferences.

Basically, the paper is written in a fully self-contained manner. All mathematical concepts and results needed in the text will be found in the next section. In the last part of Section 2, we will present a Fatou's lemma (Theorem 1) for correspondences (multifunctions). Since we define the resource feasibility by the Gel'fand integral, the lemma will be proved for the Gel'fand integral. Among the aforementioned papers, Yannelis [30] worked with the Bochner integrable correspondences, Cornet and Médécin [9] worked with the Gel'fand integrable mappings (functions). Finally, Balder and Sambucini [5] proved their theorem (lemma) for the Gel'fand integrable correspondences. Our result is far more modest than theirs, since we will assume that the ranges of the correspondences are contained in a fixed weak* compact set. But our proof will be much simpler than theirs and elementary. It is based upon the Hahn-Banach separation theorem (Fact 4) and Khan's extension of the Liapounoff's convexity theorem (Fact 11), hence we can avoid the advanced techniques of the convex analysis, which were extensively used by Balder and Sambucini, and the result will be enough to our purpose, or the application to prove the equilibrium existence theorem. Indeed, it will provide a natural proof for the existence theorem, which is much simpler than those of Bewley [8] or Noguchi [20]. The market model will be presented in Section 3. Section 4 will be devoted to the proofs.

2. An Infinite Dimensional Fatou's Lemma

As stated in Introduction, the commodity space of the economy in this paper is set to be

$$\ell^\infty = \{\xi = (\xi^t) | \sup_{t \geq 1} |\xi^t| < +\infty\},$$

the space of the bounded sequences with the supremum norm. It is well known that the space ℓ^∞ is a Banach space with respect to the norm $\|\xi\| = \sup_{t \geq 1} |\xi^t|$ for $\xi \in \ell^\infty$ (Royden [22]). Let $\mathbf{0} = (0, 0, \ldots)$. For

$\xi = (\xi^t) \in \mathbb{R}^\ell$ or ℓ^∞, $\xi \geq \mathbf{0}$ means that $\xi^t \geq 0$ for all t and $\xi > 0$ means that $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$. $\xi \gg \mathbf{0}$ means that $\xi^t > 0$ for all t. Finally, for $\xi = (\xi^t) \in \ell^\infty$, we denote by $\xi \ggg \mathbf{0}$ if and only if there exists an $\epsilon > 0$ such that $\xi^t \geq \epsilon$ for all t.

It is a well known fact that the dual space of ℓ^∞ is the space of bounded and finitely additive set functions on $\mathbb{N}$, which is denoted by ba,

$$ba = \left\{\pi : 2^{\mathbb{N}} \to \mathbb{R} \sup_{E \subset \mathbb{N}} |\pi(E)| < +\infty, \pi(E \cup F) = \pi(E) + \pi(F) \text{whenever } E \cap F = \emptyset\right\}.$$

Then we can show that the space ba is a Banach with the norm

$$\|\pi\| = \sup\left\{\sum_{i=1}^{n} |\pi(E_i)| E_i \cap E_j = \emptyset \text{ for } i \neq j,\ n \in \mathbb{N}\right\}.$$

Since the commodity vectors are represented by sequences, it is more natural to consider the price vectors also as sequences rather than the set functions. Therefore, the subspace ca of ba,

$$ca = \left\{\pi \in ba\ \pi\left(\cup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \pi(E_n) \text{ whenever } E_i \cap E_j = \emptyset\,(i \neq j)\right\},$$

which is the space of the bounded and countably additive set functions on $\mathbb{N}$ is more appropriate as the price space. Indeed, it is easy to see that the space ca is isometrically isomorphic to the space ℓ^1, the space of all summable sequences

$$\ell^1 = \left\{\boldsymbol{p} = (p^t) \,\middle|\, \sum_{t=1}^{\infty} |p^t| < +\infty\right\},$$

which is a separable Banach space with the norm $\|\boldsymbol{p}\| = \sum_{t=1}^{\infty} |p^t|$. It is also well known that the norm dual of ℓ^1 is ℓ^∞, or $(\ell^1)^* = \ell^\infty$, where L^* is the space of the continuous linear functionals on a normed vector space

L. Then the value of a commodity $\xi = (\xi^t) \in \ell^\infty$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is given by the natural "inner product" $\boldsymbol{p}\xi = \sum_{t=1}^\infty p^t \xi^t$.

The set function $\pi \in ba$ is called purely finitely additive if $\rho = 0$ whenever $\rho \in ca$ and $0 \le \rho \le \pi$. The relation between the ba and ca is made clear by the next fundamental theorem.

Fact 1 (Yosida-Hewitt [33]). If $\pi \in ba$ and $\pi \ge 0$, then there exist set functions $\pi_c \ge 0$ and $\pi_p \ge 0$ in ba such that π_c is countably additive and π_p is purely finitely additive and satisfy $\pi = \pi_c + \pi_p$. This decomposition is unique.

On the space ℓ^∞, we can consider the several topologies. One is of course the norm topology τ_{norm}, which was explained above. It is the strongest topology among the topologies which appear in this paper.

The weakest topology in this paper is the product topology τ_d, which is induced from the metric

$$d(\xi, \zeta) = \sum_{t=1}^{\infty} \frac{|\xi^t - \zeta^t|}{2^t(1 + |\xi^t - \zeta^t|)} \text{ for } \xi = (\xi^t), \quad \zeta = (\zeta^t) \in \ell^\infty.$$

The product topology is nothing but the topology of coordinate-wise convergence, or $\xi = (\xi^t) \to \mathbf{0}$ if and only if $\xi^t \to 0$ for all $t \in \mathbb{N}$.

A net (ξ_α) on ℓ^∞ is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(\ell^\infty, \ell^1)$-topology if and only if $\boldsymbol{p}\xi_\alpha \to 0$ for each $\boldsymbol{p} \in \ell^1$. The weak* topology is characterized by the weakest topology on ℓ^∞, which makes $(\ell^\infty)^* = \ell^1$. Then it is stronger than the product topology, since the latter is characterized by $\xi_\alpha \to 0$ if and only if $\boldsymbol{e}_t\xi_\alpha \to 0$ for all for each

$\boldsymbol{e}_t = (0 \dots 0, 1, 0 \dots) \in \ell^1$, where 1 is at the t-th coordinate. A subbase of the neighbourhood system of $\mathbf{0} \in \ell^\infty$ in the weak* topology is the family of the sets U of the form

$$U = \left\{\xi \in \ell^\infty \,\middle|\, |\boldsymbol{p}\xi| < \epsilon\right\}, \quad \epsilon > 0, \quad \boldsymbol{p} \in \ell^1.$$

The strongest topology on ℓ^∞ which makes $(\ell^\infty)^* = \ell^1$ is called the Mackey topology $\tau(\ell^\infty, \ell^1)$. It is characterized by saying that a net (ξ_α) on ℓ^∞ is said to converge to $\mathbf{0}$ in $\tau(\ell^\infty, \ell^1)$-topology if and only if $\sup\{|\boldsymbol{p}\xi_\alpha| \,|\, \boldsymbol{p} \in C\} \to 0$ on every $\sigma(\ell^1, \ell^\infty)$-compact, convex, and circled subset C of ℓ^1, where a set C is circled if and only if $rC \subset C$ for $-1 \le r \le 1$, and the topology $\sigma(\ell^1, \ell^\infty)$ is defined analogously as $\sigma(\ell^\infty, \ell^1)$, namely, that a net $(\boldsymbol{p}_\alpha)$ on ℓ^1 is said to converge to $\mathbf{0}$ in the $\sigma(\ell^1, \ell^\infty)$-topology if and only if $\boldsymbol{p}_\alpha\xi \to 0$ for each $\xi \in \ell^\infty$. The topology $\tau(\ell^\infty, \ell^1)$ is weaker than the norm topology. Hence, we have $\tau_d \subset \sigma(\ell^\infty, \ell^1) \subset \tau(\ell^\infty, \ell^1) \subset \tau_{norm}$.

Similarly, a net (π_α) on ba is said to converge to $\mathbf{0}$ in the weak* topology or $\sigma(ba, \ell^\infty)$-topology if and only if $\pi_\alpha\xi \to 0$ for each $\xi \in \ell^\infty$.

In general, let L be a normed vector space and L^* its dual space. The dual norm on L^* is given by $\|\boldsymbol{p}\| = \sup\{|\boldsymbol{p}\xi| \,|\xi \in L, \|\xi\| \le 1\}$. A net (ξ_α) in L converges to $\xi \in L$ in the $\sigma(L, L^*)$-topology or weak topology if and only if $\pi\xi_\alpha \to \pi\xi$ for every $\pi \in L^*$. A net (π_α) in L^* converges to $\pi \in L^*$ in the $\sigma(L^*, L)$-topology or weak* topology if and only if $\pi_\alpha\xi \to \pi\xi$ for every $\xi \in L$.

We can use the next useful proposition on bounded subsets of the space ℓ^{∞}.

Fact 2 (Bewley [8], p.226)**.** Let Z be a (norm) bounded subset of ℓ^{∞}. Then on the set Z, the Mackey topology $\tau(\ell^{\infty}, \ell^{1})$ coincides with the product topology τ_{d}.

Bounded subsets of ℓ^{∞} are $\sigma(\ell^{\infty}, \ell^{1})$-weakly compact, namely, that the weak*-closure of the sets are weak*-compact by the Banach-Alaoglu's theorem.

Fact 3 (Rudin [23], pp.68-70)**.** Let L be a Banach space. The unit ball of L^{*}, $B = \{\pi \in L^{*} \mid \|\pi\| \leq 1\}$ is compact in the $\sigma(L^{*}, L)$-topology. Moreover, if L is a separable Banach space, then bounded subsets of L^{*} is compact and metrizable in the weak* topology.

These are examples of the locally convex topological vector space. It is defined as a vector space endowed with the compatible topology (vector space operations are continuous with respect to this topology), whose every neighbourhood of **0** includes a convex neighbourhood of **0**. The dual space of a topological vector space L is also denoted by L^{*}. Let A be a subset of a locally convex topological vector space. We denote by $co(A)$ the convex hull of A. $\overline{co}(A) \equiv \overline{co(A)}$ is its closure. The famous Minkowski's separation hyper-plane theorem is extended to locally convex topological vector spaces.

Fact 4 (Aliprantis and Border [1], p.147)**.** For disjoint nonempty convex subsets A and B of a locally convex L, if one is compact and the other closed, then there is a nonzero continuous functional $\boldsymbol{p} \in L^{*}$ such that for some $\epsilon > 0$, $\boldsymbol{qx} \leq 0$ for all $\boldsymbol{x} \in A$ and $\boldsymbol{qy} \geq \epsilon$ for all $\boldsymbol{y} \in B$.

Let X be a complete and separable metric space. Let F_{n} be a sequence of subsets of X. The topological limes superior $Ls(F_{n})$ is defined by

$\xi \in Ls(F_n)$ if and only if there exists a sub-sequence $F_{n(k)}$ with $\xi_{n(k)} \in F_{n(k)}$ for all k and $\xi_{n(k)} \to \xi (k \to \infty)$.

Let X be a complete separable metric (or normed) space. The Borel σ-algebra of X, which is defined as the σ-algebra generated by open subsets of X is denoted by $\mathcal{B}(X)$. For a separable Banach space L, let $B_k = \{\pi \in L^* \mid \|\pi\| \leq k\}$. Then B_k are compact and metrizable in the weak* topology by Fact 3. Hence $L^* = \bigcup_{k=1}^{\infty} B_k$ is a Suslin space. Therefore, we have (Balder and Sambucini [5], p.384).

Fact 5. The Borel σ-algebra $\mathcal{B}(L^*)$ is the same for the weak* topology and the norm topology for a separable Banach space L.

Let $(A, \mathcal{A}, \nu)$ be a complete and finite measure space. Aumann [3] proved

Fact 6. Let $\phi : A \to X$ be a nonempty valued correspondence with a measurable graph, or $\{(a, \boldsymbol{x}) \in A \times X | \boldsymbol{x} \in \phi(a)\} \in \mathcal{A} \times \mathcal{B}(X)$. Then there exists a measurable function $f : A \to X$ such that $f(a) \in \phi(a)$ a.e. and the next theorem is known as Kuratowski and Ryll-Nardzewski measurable selection theorem.

Fact 7 (Yannelis [31], p.40)**.** Let $\phi : A \to X$ be a closed non-empty valued correspondence, which is lower measurable, or $\{a \in A | \phi(a) \cap U \neq \emptyset\} \in \mathcal{A}$ for every open subset U of X. Then, there exists a measurable function $f : A \to X$ such that $f(a) \in \phi(a)$ a.e.

Consider a sequence of correspondences $\phi_n : A \to X$. For each $a \in A$, if $Ls(\phi_n(a)) \neq \emptyset$, we can define a map $Ls(\phi_n(\cdot)) : A \to X$, $a \mapsto Ls(\phi_n(a))$. Then we have

Fact 8 (Noguchi [19], p.278)**.** Let $\phi_n : A \to X$ be a sequence of nonempty valued lower measurable correspondences. Then the map $Ls(\phi_n(\cdot))$ has a measurable graph.

A map $f : A \to \ell^{\infty}$ is said to be weak*-measurable if for each $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}f(a)$ is measurable. A map $f : A \to \ell^{\infty}$ is weakly measurable, if for each $\pi \in ba$, $\pi f(a)$ is measurable. A weak*-(weakly) measurable map $f(a)$ is said to be Gel'fand (Pettis) integrable, if there exists an element $\xi \in \ell^{\infty}$ such that for each $\boldsymbol{p} \in \ell^1 (\pi \in ba)$, $\boldsymbol{p}\xi = \int \boldsymbol{p}f(a)d\nu\, (\pi\xi = \int \pi f(a)d\nu)$. The vector ξ is denoted by $\int f(a)d\nu$ and called Gel'fand (Pettis) integral of *f*. It is obvious that if a map *f* is Pettis integrable, it is Gel'fand integrable and the both integrals coincide.

Fact 9 (Diestel and Uhl [11], p.53-54)**.** If $f : A \to \ell^{\infty}$ is weak*-(weakly) measurable and $\boldsymbol{p}f(a)(\pi f(a))$ is integrable function for all $\boldsymbol{p} \in \ell^1(\pi \in ba)$, then *f* is Gel'fand (Pettis) integrable.

Fact 10. Let $\{f_n\}$ be a sequence of Gel'fand integrable functions from A to ℓ^{∞}, which converges a.e. to *f* in the weak* topology. Then, it follows that $\int_A f_n(a)d\nu \to \int_A f(a)d\nu$ in the weak* topology.

Proof. Let $\boldsymbol{p} \in \ell^1$. Then we have

$$\boldsymbol{p}\int_A f_n(a)d\nu = \int_A \boldsymbol{p}f_n(a)d\nu \to \int_A \boldsymbol{p}f(a)d\nu = \boldsymbol{p}\int_A f(a)d\nu,$$

hence $\int_A f_n(a)d\nu \to \int_A f(a)d\nu$ in the $\sigma(\ell^{\infty}, \ell^1)$-topology. ∎

The next theorem is due to Ali Khan [15], which is an infinite dimensional generalization of the Liapounoff's theorem.

Fact 11 (Yannelis [30], p.30)**.** Let $(A, \mathcal{A}, \nu)$ be a complete and finite measure space, and L^* be the dual space of a separable Banach space L. Let $\phi : A \to L^*$ be a correspondence with a weak*-measurable graph, or $\{(a, \boldsymbol{x}) \in A \times X | \ \boldsymbol{x} \in \phi(a)\} \in \mathcal{A} \times \mathcal{B}(L^*)$, where $\mathcal{B}(L^*)$ is the family of Borel measurable subsets of L^* in the weak* topology. Assume that

$\phi(a)$ is weak*-closed and bounded for all $a \in A$. Then, it follows that

$$\overline{\int_B \phi(a) d\nu} = \int_B \overline{co}(\phi(a)) d\nu \text{ for all } B \in \mathcal{A},$$

where the closure is taken with respect to the weak* topology. Moreover, $\int_B \overline{co}(\phi(a)) d\nu$ is convex and compact in the weak* topology for every $B \in \mathcal{A}$.

The next fact is well known for mathematical economics as Fatou's lemma for finite dimensional spaces (the last equality is known as Liapounoff's theorem).

Fact 12 (Hildenbrand [13], Theorem 6, p.68)**.** Let $(\phi)_{n \in \mathbb{N}}$ be a sequence of measurable correspondences of a measure space $(A, \mathcal{A}, \nu)$ to $\mathbb{R}_+^\ell$ such that there exists a sequence $g_n(a)$ of functions A to $\mathbb{R}_+^\ell$ satisfying (i) $\phi_n(a) \leq g_n(a)$ a.e. on A, and (ii) the sequence $g_n(a)$ is uniformly integrable and the set $\{g_n(a) \| n = 1, 2, \dots\}$ is bounded a.e. on A. Then $Ls(\int_A \phi_n(a) d\nu) \subset \int_A (Ls(\phi_n(a)) d\nu = \int_A co(Ls(\phi(a))) d\nu$.

We need the next theorem which is an infinite dimensional version of Fact 12. For the corresponding theorem for Bochner integrals, see Yannelis ([30], Theorem 6.5, p.23).

Theorem 1. *Let L^* be the dual space of a separable Banach space L. Let $\phi_{n \in \mathbb{N}}$ be a sequence of weak*-closed valued and lower measurable correspondences of a complete and finite measure space $(A, \mathcal{A}, \nu)$ to a (norm) bounded subset X of L^*. Then*

$$Ls\left(\int_A \phi_n(a) d\nu\right) \subset \overline{\int_A Ls(\phi_n(a)) d\nu},$$

where the closure is taken with respect to the weak* topology.

The proof of Theorem 1 will be given in Section 4.

3. A General Equilibrium Model

Let $\beta > 0$ be a given positive number. We will assume that the consumption set X of each consumer is the set of nonnegative vectors, whose coordinates are bounded by β,

$$X = \{\xi = (\xi^t) \in \ell^\infty | \, 0 \leq \xi^t \leq \beta \text{ for } t \geq 1\}.$$

X is the set of all admissible consumption vectors. Then of course $\beta > 0$ is intended to be a very large number. From Fact 2, we have $\tau_d = \sigma(\ell^\infty, \ell^1) = \tau(\ell^\infty, \ell^1)$ on the set X. Since X is compact in τ_d (hence $\sigma(\ell^\infty, \ell^1)$ and $\tau(\ell^\infty, \ell^1)$) topology, it is complete and separable metric space.

As usual, a preference $\succsim$ is a complete, transitive, and reflexive binary relation on X. We denote $(\xi, \zeta) \in \succsim$ by $\xi \succsim \zeta$. The interpretation is that the consumption vector ξ is at least as desired as ζ. $\xi \prec \zeta$ means that $(\xi, \zeta) \notin \succsim$. Let $\mathcal{P} \subset 2^{X \times X}$ be the collection of allowed preference relations.

Each consumer has a vector $\omega \in \ell^\infty$, which is called an initial endowment vector owned by the consumer before trading in the market. We denote the set of all endowment vectors by Ω and assume that it is of the form

$$\Omega = \{\omega = (\omega^t) \in \ell^\infty | \, 0 \leq \omega^t \leq \gamma \text{ for } t \geq 1\},$$

for some $\gamma > 0$. We then assume that $\gamma < \beta$. The set Ω is also a compact metric space by the same reason as the space X.

Let $(A, \mathcal{A}, \nu)$ be a probability space of consumers, which is assumed to be complete and atomless. Each element $a \in A$ stands for the "name" of the consumers. In other words, a consumer is identified by an element of A. For each $a \in A$, we have a map which assigns a his/her preference $\succsim_a \in \mathcal{P}$ and is measurable in the sense that

$$\{(a, \xi, \zeta) \in A \times X \times X | \xi \succsim_a \zeta\} \in \mathcal{A} \times \mathcal{B}(X) \times \mathcal{B}(X),$$

where $\mathcal{B}(X)$ is the set of Borel subsets of X in the weak* topology.

An endowment assignment map ω is a Borel measurable map from A to $(\Omega, \mathcal{B}(\Omega))$, $a \mapsto \omega(a) \in \Omega$, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra with respect to the weak* topology. Then it is weak*-measurable, or for every $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}\omega(a) : A \to \mathbb{R}$ is a measurable function. Note that by Fact 9 in Section 2 combined with the assumption for Ω, the map ω is Gel'fand integrable. Since the Borel σ-algebra with respect to the weak* topology is the same as that in the norm topology (Fact 5), then all σ-algebras with respect to the weak*, weak, and the norm topology coincide, since $\sigma(\ell^\infty, \ell^1) \subset \sigma(\ell^\infty, ba) \subset \tau_{norm}$. Therefore, a map ω is also weakly measurable, hence it is Pettis integrable by Fact 9.

The definition of the economy follows Aumann [2, 4] and Hildenbrand [13].

Definition 1. An economy $\mathcal{E}$ is a mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega$ defined by $a \mapsto (\succsim_a, \omega(a))$.

The assumptions for the preferences are:

Assumption (PR)

(i) $\succsim \in \mathcal{P}$ is complete, transitive, and reflexive, which is closed in $X \times X$ in the weak* topology.

(ii) (Monotonicity). For each $\xi \in X$ and $\zeta \in X$ such that $\xi < \zeta$, $\xi \prec \zeta$.

(iii) (Convexity). For $\succsim \in \mathcal{P}$, the set $\{\xi \in X | \xi \succsim \zeta\}$ is convex for all $\zeta \in X$.

An integrable map $\xi : A \to X$ is called an allocation. The allocation ξ is said to be feasible if $\int_A \xi(a)d\nu \leq \int_A \omega(a)d\nu$. It is said to be exactly feasible, if $\int_A \xi(a)d\nu = \int_A \omega(a)d\nu$. Note that the Gel'fand integral $\int_A \xi(a)d\nu$ also exists if ξ is weak*-measurable by virtue of Fact 9.

The next equilibrium concept is due to Debreu [10].

Definition 2. A pair $(\boldsymbol{p}, \xi)$ of a price vector $\boldsymbol{p} \in \ell_+^1$ with $\boldsymbol{p} \neq \boldsymbol{0}$ and an allocation $\xi : A \to X$ is called a quasi-competitive equilibrium of the economy μ, if the following conditions hold:

(Q-1) $\boldsymbol{p}\xi(a) \leq \boldsymbol{p}\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\boldsymbol{p}\zeta < \boldsymbol{p}\omega(a)$ a.e..

(Q-2) $\int_A \xi(a)d\nu \leq \int_A \omega(a)d\nu.$

The condition (Q-1) says that the quasi-demand condition is met, or the vector ξ is maximal among the vectors, which are strictly cheaper than the endowment vectors. The condition (Q-2) says that the allocation ξ is a feasible one. The quasi-equilibrium is slightly weaker than the competitive equilibrium, by which the exact equilibrium concept is realized.

Definition 3. A pair $(\boldsymbol{p}, \xi)$ of a price vector $\boldsymbol{p} \in \ell_+^1$ with $\boldsymbol{p} \neq \boldsymbol{0}$ and an allocation $\xi : A \to X$ is called a competitive equilibrium of the economy $\mathcal{E}$, if the following conditions hold:

(E-1) $\boldsymbol{p}\xi(a) \leq \boldsymbol{p}\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\boldsymbol{p}\zeta \leq \boldsymbol{p}\omega(a)$ a.e..

(E-2) $\int_A \xi(a)d\nu = \int_A \omega(a)d\nu.$

The condition (E-1) says that almost all consumers maximize their utilities under their budget constraints. The condition (E-2) says that the total equilibrium allocation is exactly equal to the total endowment (exactly feasible).

The following assumption on the total endowment, which means that every commodity is available in the market is standard.

Assumption (TP) (Positive total endowment). $\int_A \omega(a) d\nu \gg \mathbf{0}$.

In order to obtain a competitive equilibrium, however, we need a stronger assumption on the individual initial endowment.

Assumption (IP) (Individually positive endowments). $\omega(a) \gg \mathbf{0}$ a.e..

The main result of this paper now reads

Theorem 2. *Let $\mathcal{E}$ be an economy which satisfies the Assumptions (PR) and (TP). Then there exists a quasi-competitive equililbrium $(\boldsymbol{p}, \xi)$ for $\mathcal{E}$.*

As a corollary of Theorem 2, we obtain

Theorem 3. *Let $\mathcal{E}$ be an economy which satisfies the Assumptions (PR) and (IP). Then there exists a competitive equilibrium $(\boldsymbol{p}, \xi)$ for $\mathcal{E}$.*

4. Proofs of Theorems

Proof of Theorem 1. By assumption, there exists an $\alpha > 0$ with $\|\boldsymbol{x}\| \le \alpha$ for all $\boldsymbol{x} \in X$, and we can assume that $\|\phi_n(a)\| \le \alpha$ for all $a \in A$ and for all $n = 1, 2, \ldots$. Then by Fact 3, the set X is a compact metric space, hence complete and separable. Take any $\boldsymbol{p} \in L$. We claim that for any sequence F_n of nonempty subsets of X, $\boldsymbol{p}Ls(F_n) = Ls(\boldsymbol{p}F_n)$. Indeed, suppose $\boldsymbol{px} \in \boldsymbol{p}Ls(F_n)$. Then, there exists a sub-sequence $\{\boldsymbol{x}_{n_q}\}$ of $\boldsymbol{x}_n$ such that $\boldsymbol{x}_{n_q} \in F_{n_q}$ for all q, and $\boldsymbol{x}_{n_q} \to \boldsymbol{x}$, hence $\boldsymbol{px}_{n_q} \to \boldsymbol{px}$, which implies that $\boldsymbol{px} \in Ls(\boldsymbol{p}F_n)$. Conversely, suppose $w \in Ls(\boldsymbol{p}F_n)$. Then there exists a sub-sequence $\boldsymbol{x}_{n_q}$ such that $\boldsymbol{x}_{n_q} \in F_{n_q}$ for all q and $\boldsymbol{px}_{n_q} \to w$. Since X is a compact metric space, we can take a converging sub-sequence of $\boldsymbol{x}_{n_q}$ still denoted by $\boldsymbol{x}_{n_q}$ with $\boldsymbol{x}_{n_q} \to \boldsymbol{x}$ in the weak* topology, or $\boldsymbol{x} \in Ls(F_n)$. Since $\boldsymbol{px}_{n_q} \to \boldsymbol{px} = w$, we have $w \in \boldsymbol{p}Ls(F_n)$.

It follows from this and Fact 12 that

$$\begin{aligned} \boldsymbol{p}Ls\left(\int_A \phi_n(a)d\nu\right) &= Ls\left(\int_A \boldsymbol{p}\phi_n(a)d\nu\right) \subset \int_A Ls(\boldsymbol{p}\phi_n(a))d\nu \\ &= \boldsymbol{p}\int_A Ls(\phi_n(a))d\nu \subset \boldsymbol{p}\overline{\int_A Ls(\phi_n(a))d\nu} \\ &= \boldsymbol{p}\int_A \overline{co}Ls(\phi_n(a))d\nu, \end{aligned}$$

where in the last equality, we applied Fact 11. Note that $\int_A \phi_n(a)d\nu \neq \emptyset$ for each n by Fact 7 and $\int_A Ls(\phi_n(a))d\nu \neq \emptyset$ by Fact 8.

We now claim that $Ls(\int_A \phi_n(a)d\nu) \subset \overline{\int_A Ls(\phi_n(a))d\nu}$. Suppose not. Then there exists a vector $\boldsymbol{x} \in Ls(\int_A \phi_n(a)d\nu)$ with $\boldsymbol{x} \notin \overline{\int_A Ls(\phi_n(a))d\nu}$. Since $\overline{\int_A Ls(\phi_n(a))d\nu} = \int_A \overline{co}Ls(\phi_n(a))d\nu$ is compact and convex by Fact 11, it follows from Fact 4 that we can take a vector $\boldsymbol{q} \in L$ and $\epsilon > 0$ such that $\boldsymbol{qx} \leq 0$ and $\boldsymbol{qy} > \epsilon$ for all $\boldsymbol{y} \in \overline{\int_A Ls(\phi(a))d\nu}$. A contradiction. ∎

Proof of Theorem 2. Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be the economy. For each $n \in \mathbb{N}$, let K^n be the canonical projection of ℓ^∞ to $\mathbb{R}^n$, $K^n = \{\xi = (\xi^t) \in \ell^\infty | \xi = (\xi^1, \xi^2, \dots, \xi^n, 0, 0, \dots)\}$. Naturally, we can identify K^n with $\mathbb{R}^n$, or $K^n \approx \mathbb{R}^n$. We define

$$X^n = X \cap K^n, \succsim^n = \succsim \cap (X^n \times X^n), \mathcal{P}^n = \mathcal{P} \cap 2^{X^n \times X^n}, \Omega^n = \Omega \cap K^n,$$

and for every $\omega = (\omega^1, \omega^2, \dots, \omega^n, \omega^{n+1}, \dots) \in \Omega$, we denote $\omega_n = (\omega^1, \omega^2, \dots, \omega^n, 0, 0, \dots) \in \Omega^n$, the canonical projection of ω. They induce finite dimensional economies $\mathcal{E}^n : A \to \mathcal{P}^n \times \Omega^n$ defined by $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$, $n = 1, 2, \dots$. We have

Lemma 1. *$\mathcal{E}^n(a) \to \mathcal{E}(a)$ a.e.*

Proof. We show that $X^n \times X^n \to X \times X$ in the topology of closed convergence τ_c. It is clear that $Li(X^n \times X^n) \subset Ls(X^n \times X^n) \subset X \times X$. Therefore, it suffices to show that $X \times X \subset Li(X^n \times X^n)$. Let $(\xi, \zeta) = ((\xi^t), (\zeta^t)) \in X \times X$, and set $\xi_n = (\xi^1 \ldots \xi^n, 0, 0 \ldots)$ and similarly ζ_n for ζ. Then $(\xi_n, \zeta_n) \in X^n \times X^n$ for all n and $(\xi_n, \zeta_n) \to (\xi, \zeta)$. Hence $(\xi, \zeta) \in Li(X^n \times X^n)$. Then it follows that $\succsim^n = \succsim \cap (X^n \times X^n) \to \succsim$.

Obviously, one obtains $\omega_n \to \omega$ in the $\sigma(\ell^\infty, \ell^1)$-topology. Consequently, we have $\mathcal{E}^n(a) \to \mathcal{E}(a)$ a.e. on A. ∎

Lemma 2. *For each n, there exists a quasi-competitive equilibrium for the economy $\mathcal{E}^n$, or a price-allocation pair $(\pi_n, \xi_n(a))$, which satisfies*

(Q-1n) $\pi_n \xi_n(a) \leq \boldsymbol{p}\omega(a)$ *and* $\xi_n(a) \succsim_a \zeta$ *whenever* $\pi_n \zeta \leq \pi_n \omega_n(a)$ *and* $\pi_n \omega_n(a) > 0$ *a.e..*

(Q-2n) $\int_A \xi_n(a) d\nu \leq \int_A \omega_n(a) d\nu.$

Proof. See Appendix in Suzuki [29] or Khan-Yamazaki [16], Proposition 2. ∎

Since $\omega_n(a) \to \omega(a)$ a.e., we have $\int_A \omega_n(a) d\nu \to \int_A \omega(a) d\nu$ by Fact 10. Without loss of generality, we can assume that $\pi_n \mathbf{1} = \sum_{t=1}^n p_n^t = 1$ for all n, where $\pi_n = (p_n^t)$ and $\mathbf{1} = (1, 1, \ldots)$. Here, we have identified $\pi_n \in \mathbb{R}_+^n$ with a vector in ℓ_+^1, which is also denoted by π_n as $\pi_n = (\pi_n, 0, 0 \ldots)$. Since the set $\Delta = \{\pi \in ba_+ \mid \|\pi\| = \pi\mathbf{1} = 1\}$ is weak*-compact by the Alaoglu's theorem (Fact 3), we have a price vector

$\pi \in ba_+$ with $\pi\mathbf{1} = 1$ and a subnet $(\pi_{n(\alpha)})$ such that $\pi_{n(\alpha)} \to \pi$ in the $\sigma(ba, \ell^\infty)$-topology. Since $\int_A \omega_{n(\alpha)}(a)d\nu \to \int_A \omega(a)d\nu$ and Ω is a compact metric space, we can extract from $\{\omega_{n(\alpha)}\}$ a sequence $\{\omega_{n(\alpha_k)}\}$ denoted by $\{\omega_k\}$ with $\int_A \omega_k(a)d\nu \to \int_A \omega(a)d\nu$. Let ξ_k be the corresponding sequence extracted from $\xi_{n(\alpha)}$. Note that $\xi_k \equiv \xi_{n(\alpha_k)}$ is a sub-sequence of ξ_n. By Theorem 1, we have a Gel'fand integrable function $\xi : A \to X$ such that

$$\xi(a) \in \overline{co}Ls(\xi_k(a)) \subset \overline{co}Ls(\xi_{n(\alpha)}(a))\,\text{a.e.},$$

and

$$\int_A \xi(a)d\nu \leq \int_A \omega(a)d\nu.$$

We can show that $\overline{co}Ls(\xi_k(a)) \subset \overline{co}Ls(\xi_{n(\alpha)}(a))$ as follows. Let $\xi \in Ls(\xi_k(a))$ and take a neighbourhood U of ξ and $n(\alpha_0) \equiv n_0$. Since $\xi_{n(\alpha_k)}$ is a sub-sequence of ξ_n, we can take an $n(\alpha_1)$ from $n(\alpha_k)$ with $n(\alpha_1) \equiv n_1 > n_0$. Since $\xi \in Ls(\xi_k(a))$, we have for some $n(\alpha_2) \equiv n_2 > n_1$ that $\xi_{n(\alpha_2)} \in U$, hence $Ls(\xi_k(a)) \subset Ls(\xi_{n(\alpha)}(a))$. It follows from this that $\overline{co}Ls(\xi_k(a)) \subset \overline{co}Ls(\xi_{n(\alpha)}(a))$.

By the above inequality, we see that the condition (Q-2) is met. Let $P = \{a \in A | \, \pi\omega(a) > 0\}$. We now prove that

Lemma 3. $\xi(a) \prec_a \zeta$ *implies that* $\pi\omega(a) < \pi\zeta$ *a.e. on P.*

Proof. If the lemma was false, there exists $\zeta(a) = (\zeta^t(a)) \in X$ such that $\pi\zeta(a) \leq \pi\omega(a)$ and $\xi(a) \prec_a \zeta(a)$ on a subset of P with ν-positive measure. Since the preferences are continuous, we can assume without loss of generality that $\pi\zeta(a) < \pi\omega(a)$ and $\xi(a) \prec_a \zeta(a)$. Let $\zeta_n(a) =$

$(\zeta^1(a)\ldots\zeta^n(a), 0, 0\ldots)$ be the projection of $\zeta(a)$ to X^n. Since $\zeta_n(a) \to \zeta(a)$ in the $\sigma(\ell^\infty, \ell^1)$ topology, we have for sufficiently large N that $\pi\zeta_N(a) \leq \pi\zeta(a) < \pi\omega(a)$ and $\xi(a) \prec_a \zeta_N(a)$, since $\pi \geq \mathbf{0}$ and $\zeta_N(a) \leq \zeta(a)$. We claim that for every α, there exists an $\alpha_0 \geq \alpha$ such that $\xi_{n(\alpha_0)}(a) \prec_a \zeta_N(a)$ a.e. If not, we have for some α_0, $\xi_{n(\alpha)}(a) \succsim_a \zeta_N(a)$ for all $\alpha \geq \alpha_0$. Since $\xi(a) \in \overline{co}Ls(\xi_{n(\alpha)}(a))$ a.e., for each neighbourhood $U(a)$ of $\xi(a)$, there exists a vector $\sum_{i=1}^m x_i\xi_i(a)$ with $x_i \geq 0$ and $\sum_{i=1}^m x_i = 1$, $\xi_i(a) \in Ls(\xi_{n(\alpha)}(a))$, $i = 1\ldots m$, a.e.. Consequently, we can take a vector $\xi_{co}(a) \in U(a)$ of the form $\xi_{co}(a) = \sum_{i=1}^m x_i\xi_{n(\alpha_i)}(a)$ a.e.. Without loss of generality, we can assume that $\alpha_i \geq \alpha_0$, $i = 1\ldots m$, hence $\xi_{n(\alpha_i)}(a) \succsim_a \zeta_N(a)$, $i = 1\ldots m$, a.e.. Then one obtains from Assumption (PR)(iii) that $\xi_{co}(a) \succsim_a \zeta_N(a)$. Since $\xi_{co}(a)$ is arbitrarily close to $\xi(a)$, we have $\xi(a) \succsim_a \zeta_N(a)$ a.e., a contradiction.

Then it follows from $\pi_{n(\alpha)} \to \pi$ that for some α_0 with $n(\alpha_0) \equiv n_0 \geq N$, $0 \leq \pi_{n_0}\zeta_N(a) < \pi_{n_0}\omega(a) = \pi_{n_0}\omega_{n_0}(a)$, and $\xi_{n_0}(a) \prec_a \zeta_N(a)$, or $\xi_{n_0}(a) \prec_a^{n_0} \zeta_N(a)$. This contradicts the fact that $(\pi_{n_0}, \xi_{n_0}(a))$ is a quasi-equilibrium for $\mathcal{E}^{n_0}$. ∎

Let $\pi = \pi_c + \pi_p$ be the Yosida-Hewitt decomposition and denote $\pi_c = \boldsymbol{p}$. Suppose that $\pi\omega(a) > 0$ and $\xi(a) \prec_a \zeta$. Then we can assume that $\xi(a) \prec_a \zeta_n$ for n sufficiently large, where as usual ζ_n is the projection of ζ to $\mathbb{R}^n$. Hence, it follows from Lemma 3 that $\pi\zeta_n > \pi\omega(a)$ for n sufficiently large. Since π_p is purely finitely additive, $\pi_p(\{1\ldots n\}) = 0$ for each n. It follows from this and $\pi_c \geq \mathbf{0}$ that

$$\pi\zeta_n = (\pi_c + \pi_p)\zeta_n = \pi_c\zeta_n \leq \pi_c\zeta = \boldsymbol{p}\zeta,$$

since $\zeta_n \leq \zeta$. On the other hand, $\pi_p \geq \mathbf{0}$ and $\omega(a) \geq \mathbf{0}$ imply that $\pi\omega(a) = (\pi_c + \pi_p)\omega(a) \geq \pi_c\omega(a) = \boldsymbol{p}\omega(a)$, and consequently, we have $\boldsymbol{p}\zeta > \boldsymbol{p}\omega(a)$.

Summing up, we have verified that

Lemma 4. $\xi(a) \prec_a \zeta$ *implies that* $\boldsymbol{p}\omega(a) < \boldsymbol{p}\zeta$ *a.e. on* P.

Since the preferences are locally non-satiated, there exists $\zeta \in X$ arbitrarily close to $\xi(a)$ such that $\xi(a) \prec_a \zeta$, therefore, we have

$$\boldsymbol{p}\omega(a) \leq \boldsymbol{p}\xi(a) \text{ for almost all } a \in P.$$

On the other hand, for $a \in A$ with $\pi\omega(a) = 0$, one obtains that $0 \leq \boldsymbol{p}\omega(a) \leq \pi\omega(a) = 0 \leq \boldsymbol{p}\xi(a)$, since $\boldsymbol{p} \geq \mathbf{0}$ and $\omega(a), \xi(a) \geq \mathbf{0}$.

It follows from $\int_{\mathrm{A}} \xi(a)d\nu \leq \int_{\mathrm{A}} \omega(a)d\nu$ that

$$\int_A \boldsymbol{p}\xi(a)d\nu = \boldsymbol{p}\int_A \xi(a)d\nu \leq \boldsymbol{p}\int_A \omega(a)d\nu = \int_A \boldsymbol{p}\omega(a)d\nu.$$

Therefore $\boldsymbol{p}\xi(a) = \boldsymbol{p}\omega(a)$ a.e. on A. It follows from this and Lemma 4 that the condition (Q-1) is met, since it holds trivially for $a \in A \setminus P$. Therefore, $(\boldsymbol{p}, \xi)$ is a quasi-competitive equilibrium for $\mathcal{E}$. This proves Theorem 2. ∎

Note that the quasi-competitive equilibrium is not very interesting, if $\nu(P) = 0$. This situation is excluded, since $\pi\int_A \omega(a)d\nu = \int_A \pi\omega(a)d\nu$. Although the elements of the space *ba* generally do not commute with the Gel'fand integral. With the Pettis integral, however, they do. Then by Assumption (TP) and $\pi\mathbf{1} = 1$ we obtain that $\nu(P) = \nu(\{a \in A | \pi\omega(a) > 0\}) > 0$ as desired.

Proof of Theorem 3. We now assume Assumption (IP) instead of (TP). The condition (E-1) follows immediately from Lemma 4 and

Lemma 5. $\nu(P) = 1$.

Proof. Obvious from Assumption (IP) and $\pi\mathbf{1} = 1$. ∎

By Lemma 5, we have obtained that

$$\xi(a) \prec_a \zeta \text{ implies that } \boldsymbol{p}\omega(a) < \boldsymbol{p}\zeta \text{ a.e. on } A.$$

This shows that the condition (E-1) is met. Finally, since $\int_A \xi^t(a)d\nu \leq \int_A \omega^t(a)d\nu \leq \gamma < \beta$ for each t, there exists a positive amount of consumers with $\xi^t(a) < \beta$. Then by the monotonicity (PR)(ii), one obtains that $p^t > 0$ for all t, hence $\int_A \xi(a)d\nu = \int_A \omega(a)d\nu$, or the condition (E-2) is met. This completes the proof of the Theorem 3. ∎

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